Lagrange Multiplier

1. Uses:
   - The lagrange multiplier is used to find the max and min values of a function, f, subject to constraints.

2. Thms:
   1. Let $f: D \subset R^n \to R$ and $g: D \subset R^n \to R$ be of class C' and let $x_0 \in D$ and $g(x_0) = c$. Let $S = \{x \in D \mid g(x) = c\}$. Assume that $\nabla g(x_0) \neq 0$. If x_0 is an ext of f on S, then there is a real number, λ, s.t. $\nabla f(x_0) = \lambda \nabla g(x_0)$.

   2. If f, when constrained to a surface S, has a max or min at x_0, then $\nabla f(x_0) \perp S$.

3. Extreme Value Theorem (EVT):
   - Def:

     Let D be a compact (Bounded and Closed) set in R^n and let $f: D \subset R^n \to R$ be cont. Then, f has both a global max and global min on D.

4. How to Find Max and Min Values Using Lagrange Multipliers:

- General Method:
  Let $a = (a_1, a_2, \dots a_n) \in R^n$ be an ext for $f: R^n \to R$ subject to the constraint $g(x_1, x_2, \dots x_n) = c$. To find the coordinates of a, we solve the system:
  1. $\nabla f(a) = \lambda \nabla g(a)$
  2. $g(a) - c = 0$

  Note:
  1. λ is called a Lagrange Multiplier.
  2. "a" gives a constrained crit point.
  3. This process is called The Method of Lagrange Multiplier.

- Steps:
  1. Construct a new function $L: R^{n+1} \to R$ by $L(x, \lambda) = f(x) - \lambda(g(x) - c)$. L is called the Lagrange Function or the Lagrangian.

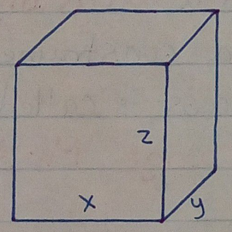
  2. Find all the crit points of L about λ and the constrained crit points of f.

3. Evaluate all the constrained crit points of f. The largest is the max value of f and the smallest is the min value of f.

5. Examples:

1. Find the dimensions of the box with the largest possible volume if the box's S. A. is $64 cm^2$.

Soln:



$$Vol = xyz$$

$$S.A. = 2xy + 2xz + 2yz = 64$$

$$\rightarrow xy + xz + yz = 32$$

$$f(x, y, z) = xyz$$

$$g(x, y, z) = xy + xz + yz = 32 = c$$

$$L = f(x,y,z) - \lambda(g(x,y,z))$$

$$= xyz - \lambda(xy + xz + yz - 32)$$

$$\frac{\partial L}{\partial x} = yz - \lambda y - \lambda z = 0 \quad (1)$$

$$\frac{\partial L}{\partial y} = xz - \lambda x - \lambda z = 0 \quad (2)$$

$$\frac{\partial L}{\partial z} = xy - \lambda x - \lambda y = 0 \quad (3)$$

$$\frac{\partial L}{\partial \lambda} = xy + xz + yz - 32 = 0 \quad (4)$$

From (1), we get: $\lambda z = yz - \lambda y$.

From (2), we get: $\lambda z = xz - \lambda x$

$$xz - \lambda x = yz - \lambda y$$
$$xz - yz = \lambda x - \lambda y$$
$$z(x-y) = \lambda(x-y)$$

Either:
1. $z = \lambda$
2. $x = y$

If $z = \lambda$, then in (1), we would have $\lambda y - \lambda y - \lambda^2 = 0$, which leads us to $\lambda = 0$, which means $z = 0$. However, if $z = 0$, the volume $= 0$. Therefore, $\lambda \neq z$.

If $x = y$, then (3) would become $y^2 - 2\lambda y = 0 \rightarrow y(y - 2\lambda) = 0$. For the same reason as above, $y \neq 0$, so $y = 2\lambda$. This means that $x = 2\lambda$.

Plugging $x=2\lambda$ into 2, we get
$2\lambda z-2\lambda^2-\lambda z=0 \rightarrow \lambda z-2\lambda^2=0.$
$\lambda(z-2\lambda)=0.$ $\lambda\neq 0$ because if it did, x, y, or z would equal 0. Therefore $z=2\lambda$.

We have:
1. $x=2\lambda$
2. $y=2\lambda$
3. $z=2\lambda$

Plugging all 3 into 4, we get:

$$3(2\lambda)^2=32$$
$$\lambda^2=\frac{32}{12}$$
$$=\frac{8}{3}$$
$$\lambda=\pm\sqrt{\frac{8}{3}}$$

$\lambda\neq-\sqrt{\frac{8}{3}}$ because if it did, then the volume would be negative.

$\therefore \lambda=\sqrt{\frac{8}{3}}$

$\therefore x=y=z=\sqrt{\frac{32}{3}}$, and the volume of the box is around $34.84\,cm^3$.

To test that the max volume of the box is $34.84 cm^3$, we find another point that satisfies the constraint and we plug it into f.

Take the point $(1, 1, \frac{31}{2})$. $f(1, 1, \frac{31}{2}) = 15.5$
< 34.84

Therefore, the dimensions that give the max vol of the box is $\sqrt{\frac{32}{3}}$ and the max vol of the box is 34.84.

2. Find the max and min of $f(x,y) = 5x - 3y$ subject to the constraint $x^2 + y^2 = 136$.

Soln:

Note: From the constraint it is clear that the region of possible solns lies on a circle of radius $\sqrt{136}$, which is closed and bounded.
I.e. $-\sqrt{136} \leq x, y \leq \sqrt{136}$
$\therefore$ By EVT, we know that a max and min value must occur.

$f(x,y) = 5x - 3y$

$g(x,y) = x^2 + y^2 = 136 = c$

$L = f - \lambda(g)$

$= 5x - 3y - \lambda(x^2 + y^2 - 136)$

$$\frac{\partial L}{\partial x} = 5 - 2\lambda x = 0 \longrightarrow x = \frac{5}{2\lambda}, \lambda \neq 0$$

$$\frac{\partial L}{\partial y} = -3 - 2\lambda y = 0 \longrightarrow y = \frac{-3}{2\lambda}, \lambda \neq 0$$

$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 - 136 = 0$$

Plugging our values of x and y into $\frac{\partial L}{\partial \lambda}$, we get: $\frac{25}{4\lambda^2} + \frac{9}{4\lambda^2} = 136$

$$\lambda^2 = \frac{34}{544}$$

$$= \frac{1}{16}$$

$$\lambda = \pm \frac{1}{4}$$

$x = \pm 10$

$y = \mp 6$

We get 2 crit points:

1. $(10,-6)$
2. $(-10,6)$

To find the max and min, we plug both points into $f(x,y)$ and see which one is bigger and which one is smaller.

$f(10,-6) = 68$ Max

$f(-10,6) = -68$ Min

3. Find the max and min values of $f(x,y) = 4x^2 + 10y^2$ on the disk $x^2+y^2 \leq 4$.

Soln:

Note: Because this is a compact set, EVT tells us that there will be a max and a min.

Furthermore, we have to split $x^2+y^2 \leq 4$ into:

1. $x^2+y^2 < 4$
2. $x^2+y^2 = 4$

Interior $(x^2+y^2<4)$:

$f_x = 8x = 0 \rightarrow x = 0$
$f_y = 20y = 0 \rightarrow y = 0$

We need to check if $(0,0)$ is within the interior.

$0^2+0^2=0$
<4, as wanted

$\therefore$ $(0,0)$ is a crit point.

Boundary $(x^2+y^2=4)$:

$f(x,y) = 4x^2+10y^2$
$g(x,y) = x^2+y^2 = 4 = c$
$L = f - \lambda g$
$= 4x^2+10y^2-\lambda(x^2+y^2-4)$

$\frac{\partial L}{\partial x} = 8x - 2\lambda x = 0 \rightarrow x(4-\lambda)=0$
$x=0$ or $\lambda=4$

$\frac{\partial L}{\partial y} = 20y - 2\lambda y = 0 \rightarrow y(10-\lambda)=0$
$y=0$ or $\lambda=10$

$\frac{\partial L}{\partial \lambda} = x^2+y^2-4=0$

If $x=0$, $y^2-4=0$

$y^2=4$

$y=\pm 2$

$(0,-2)$ and $(0,2)$ are 2 crit points.

If $y=0$, $x^2=4$

$x=\pm 2$

$(-2,0)$ and $(2,0)$ are 2 crit points.

If $\lambda=4$, $20y-2(4)y=0$

$12y=0$

$y=0$

We already covered this.

If $\lambda=$ 10, $8x-20x=0$

$-12x=0$

$x=0$

We already covered this.

In total, we have 5 crit points:

1. $(0,0)$	$f(0,0)=0$	Min
2. $(0,-2)$	$f(0,-2)=40$	Max
3. $(0,2)$	$f(0,2)=40$	Max
4. $(2,0)$	$f(2,0)=16$	
5. $(-2,0)$	$f(-2,0)=16$	

4. Find the max and min of $f(x,y,z) = 4y - 2z$ subject to the constraints $2x - y - z = 2$ and $x^2 + y^2 = 1$.

Soln:

Note: Here, there are 2 constraints.

Furthermore, because x and y are bounded; $-1 \le x, y \le 1$; z is also bounded. Therefore, by EVT, there must be a max and a min.

$$f(x,y,z) = 4y - 2z$$
$$g_1(x,y,z) = 2x - y - z = 2 = c$$
$$g_2(x,y,z) = x^2 + y^2 = 1 = d$$
$$L = f - \lambda_1(g_1) - \lambda_2(g_2)$$
$$= 4y - 2z - \lambda_1(2x - y - z - 2) - \lambda_2(x^2 + y^2 - 1)$$

$$L_x = -2\lambda_1 - 2\lambda_2 x = 0 \rightarrow \lambda_1 = -\lambda_2 x$$
$$L_y = 4 + \lambda_1 - 2\lambda_2 y = 0 \rightarrow \lambda_1 = 2\lambda_2 y - 4$$
$$L_{\lambda_1} = 2x - y - z - 2 = 0$$
$$L_{\lambda_2} = x^2 + y^2 - 1 = 0$$
$$L_z = -2 + \lambda_1 = 0 \rightarrow \lambda_1 = 2$$

$x = -\frac{2}{\lambda_2}$

$2 = 2\lambda_2 y - 4$

$6 = 2\lambda_2 y$

$3 = \lambda_2 y$

$y = \frac{3}{\lambda_2}$

Plugging the values of x and y into $L\lambda_2$, we get:

$\left(-\frac{2}{\lambda_2}\right)^2 + \left(\frac{3}{\lambda_2}\right)^2 = 1$

$\frac{13}{(\lambda_2)^2} = 1$

$(\lambda_2)^2 = 13$

$\lambda_2 = \pm\sqrt{13}$

$x = \mp\frac{2}{\sqrt{13}}, \quad y = \pm\frac{3}{\sqrt{13}}$

Plugging in the values of x and y into $L\lambda_1$, we get:

1. $2\left(-\frac{2}{\sqrt{13}}\right) - \left(\frac{3}{\sqrt{13}}\right) - z = 2$

$-\frac{7}{\sqrt{13}} - z = 2$

$z = -\left(2 + \frac{7}{\sqrt{13}}\right)$

2. $2\left(\frac{2}{\sqrt{13}}\right) - \left(-\frac{3}{\sqrt{13}}\right) - z = 2$

$$\frac{7}{\sqrt{13}} - z = 2$$

$$z = -\left(2 - \frac{7}{\sqrt{13}}\right)$$

The 2 critical points found are:

1. $\left(-\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}, -2-\frac{7}{\sqrt{13}}\right) \rightarrow f(\ldots) = 11.2$ Max

2. $\left(\frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}}, -2+\frac{7}{\sqrt{13}}\right) \rightarrow f(\ldots) = -3.2$ Min